Example 7: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. If A is invertible, calculate A^{-1}. If not, explain why.

$$[A \mid I_n] \sim [I_n \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\substack{R_3 := R_3 - R_1 \\ R_2 := R_2 - R_1}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R_2 := \tfrac{-1}{2} R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 1/2 & -1/2 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] \underset{\sim}{R_3 := R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & -3/2 & -1/2 & -1/2 & 1 \end{array}\right]$$

$$\underset{\sim}{R_3 := -\tfrac{2}{3} R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & -2/3 \end{array}\right] \underset{\sim}{\substack{R_1 := R_1 - R_3 \\ R_2 := R_2 - \frac{1}{2} R_3}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2/3 & -1/3 & 2/3 \\ 0 & 1 & 0 & 1/3 & -2/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & -2/3 \end{array}\right] \sim$$

$$\underset{\sim}{R_1 := R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & -2/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & -2/3 \end{array}\right] = [I_3 \mid A^{-1}]$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Exercise:

check $AA^{-1} = I_3$